

IRREDUCIBLE ULRICH BUNDLES ON ISOTROPIC GRASSMANNIANS

ANTON FONAREV

ABSTRACT. We classify irreducible equivariant Ulrich vector bundles on isotropic Grassmannians.

1. INTRODUCTION

Ulrich bundles were introduced in [5] in order to study Chow forms. The notion of an Ulrich module appeared much earlier in commutative algebra. Ulrich himself gave a certain sharp upper bound on the minimal number of generators for a maximal Cohen–Macaulay module over a Cohen–Macaulay homogeneous ring. Ulrich modules are precisely those for which the upper bound is attained (see [9]). There are several equivalent definitions of Ulrich bundles. Though we are going to use a less enlightening cohomological one, it might be motivating to formulate the most geometric and, to our taste, the easiest one. Given a d -dimensional projective variety $X \subset \mathbb{P}^N$, an *Ulrich bundle* on X is a vector bundle E such that π_*E is a trivial bundle for a general linear projection $\pi : X \rightarrow \mathbb{P}^d$. Further details may be found in [5].

It was asked in [5] whether every projective variety admits an Ulrich bundle. So far the answer is known in few cases which include hypersurfaces and complete intersections [7], del Pezzo surfaces [5], and abelian surfaces [1].

While it is not clear whether the answer to the general question is positive, in the case of rational homogeneous varieties one has a large test class of vector bundles, namely equivariant ones. In [4] the authors fully classified irreducible equivariant Ulrich vector bundles on Grassmannians over an algebraically closed field of characteristic zero. These results were further improved in [3], where most of partial flag varieties were treated. In the current paper we move in an orthogonal (or symplectic) direction and classify irreducible equivariant Ulrich bundles on isotropic Grassmannians, that is varieties of the form G/P , where G is a classical group of type B_n , C_n , or D_n , and P is a maximal parabolic subgroup.

The following meta theorem is the main result of the present paper.

Theorem. *The only isotropic Grassmannians admitting an equivariant irreducible Ulrich vector bundle are the symplectic Grassmannians of planes $\mathrm{IGr}(2, 2n)$ for $n \geq 2$, odd and even-dimensional quadrics Q^d , orthogonal Grassmannians of planes $\mathrm{OGr}(2, m)$ for $m \geq 4$, even orthogonal Grassmannians of 3-spaces $\mathrm{OGr}(3, 4q + 6)$ for $q \geq 0$, and $\mathrm{OGr}(4, 8)$. In each case the corresponding bundles are classified.*

The paper is organized as follows. In Section 2 we collect some preliminary definitions and provide a simple criterion for an equivariant irreducible vector bundle to be Ulrich. In Section 3 we treat all isotropic Grassmannians in type C_n but maximal (Lagrangian) ones. Similarly, in sections 4 and 5 we deal with all but maximal Grassmannians in types B_n and D_n respectively. Finally, Section 6 is devoted to maximal Grassmannians, both symplectic and orthogonal.

The author is grateful to the anonymous reviewer for careful reading of the original manuscript.

This work is supported by the RSF under a grant 14-50-00005.

2. PRELIMINARIES

Let G be a simple algebraic group of type B_n , C_n , or D_n over an algebraically closed field of characteristic zero. We fix a maximal torus $T \subset G$ along with a Borel subgroup $B \supset T$. Let P_G and W denote the weight lattice and the Weil group of G respectively. In the following all the roots and weights will be expressed in terms of the standard orthonormal basis $\langle e_1, e_2, \dots, e_n \rangle$ in a fixed vector space \mathbb{R}^n . Denote by $\Phi \supset \Phi^+ \supset \Delta$ the root system of G , the set of positive roots, and the set of simple roots respectively. We use the standard numbering of the simple roots $\Delta = \{\varepsilon_1, \dots, \varepsilon_n\}$.

Our goal is to study vector bundles on varieties of the form $X = G/P$, where $P = P_k$ is the maximal parabolic subgroup associated with the set $\Delta \setminus \{\varepsilon_k\}$. We call such varieties *isotropic Grassmannians*. In particular, maximal isotropic Grassmannians correspond to $\Delta \setminus \{\varepsilon_n\}$.

It is well known that the category of G -equivariant vector bundles on X is equivalent to the category of representations of P :

$$\mathrm{Coh}^G(G/P) \simeq \mathrm{Rep} - P,$$

see [2]. Moreover, this is an equivalence of tensor abelian categories.

The group P is not reductive, however, we are interested only in irreducible equivariant vector bundles on X . Those correspond to irreducible representations of P , which in turn are all induced from the reductive Levi quotient group $L = P/U$ (here U denotes the unipotent radical of P).

The weight lattice P_L is canonically isomorphic to P_G . Let $P_L^+ \subset P_L$ denote the cone of L -dominant weights. It is generated by $\omega_1, \dots, \omega_n$ and $-\omega_k$. For a given $\lambda \in P_L^+$ we denote by \mathcal{U}^λ the corresponding irreducible equivariant vector bundle on X . In particular, \mathcal{U}^{ω_k} is the ample generator of $\mathrm{Pic} X \simeq \mathbb{Z}$.

The Borel–Bott–Weil theorem is the tool we use to compute cohomology of irreducible vector bundles on Grassmannians (see, e.g., [10]). Let ρ denote the half-sum of positive roots of G , which also equals the sum of fundamental weights. Recall that a weight is called *singular* if it lies on a wall of a Weyl chamber, i.e. is fixed by a nontrivial element of W_G . Equivalently, a weight is singular if and only if it is orthogonal to some (positive) root of G .

Theorem 2.1 (Borel–Bott–Weil). *Let $\lambda \in P_L^+$. If $\lambda + \rho$ is singular, then*

$$H^\bullet(X, \mathcal{U}^\lambda) = 0.$$

Otherwise, $\lambda + \rho$ lies in the interior of some Weyl chamber, and there exists a unique element $w \in W$ such that $w(\lambda + \rho) \in P_G^+$. Let l denote the length of w . Then the only nontrivial cohomology

$$H^l(X, \mathcal{U}^\lambda) = V^{w(\lambda+\rho)-\rho}$$

is an irreducible representation of G of highest weight $w(\lambda + \rho) - \rho$.

Now that we have all necessary information about irreducible equivariant vector bundles on isotropic Grassmannians, we can turn to Ulrich bundles. The latter were introduced in [5] where the authors study Chow forms. We are going to use the original cohomological characterization.

Definition 2.2. Let $X \subset \mathbb{P}^N$ be a projective variety of dimension d . A vector bundle \mathcal{E} on X is called *Ulrich* if $H^i(X, \mathcal{E}(t)) = 0$ for $0 < i < d$ and all t , while $H^0(X, \mathcal{E}(t)) = 0$ for $t < 0$, and $H^d(X, \mathcal{E}(t)) = 0$ for $t \geq -d$.

The goal of the present paper is to classify all irreducible equivariant Ulrich bundles on $X = G/P_{\varepsilon_k}$ under the natural embedding given by \mathcal{U}^{ω_k} . Let $d = \dim X$. We are now going to reduce the problem to a representation-theoretic one. For a given L -dominant weight $\lambda \in P_L^+$ define

$$\mathrm{lrr}(\lambda) = \{t \in \mathbb{Z} \mid \lambda + \rho - t\omega_k \text{ is singular}\}.$$

Lemma 2.3. *The set $\mathrm{lrr}(\lambda)$ contains at most d elements.*

Proof. The weight $\lambda = \sum x_i \omega_i$ is L -dominant if and only if $x_i \geq 0$ for all $i \neq k$. Thus, for $\lambda \in P_{\mathbb{L}}^+$ the coefficients y_i in the decomposition $\lambda + \rho = \sum y_i \omega_i$ are positive for all $i \neq k$. We are interested in all $t \in \mathbb{Z}$ for which $\lambda + \rho - t\omega_k$ is orthogonal to a positive root of \mathbf{G} . Put

$$\Phi' = \{\alpha \in \Phi^+ \mid \alpha = \sum_{i \neq k} p_i \varepsilon_i\}.$$

For any $\alpha \in \Phi'$ one has $(\alpha, \omega_k) = 0$, thus $(\alpha, \lambda + \rho - t\omega_k) = \sum_{i \neq k} y_i (\alpha, \omega_i) > 0$ for any t . Meanwhile, for any $\alpha \in \Phi^+ \setminus \Phi'$ one has $(\alpha, \omega_k) > 0$. Thus, there exists a unique *rational* t such that $(\alpha, \lambda + \rho - t\omega_k) = 0$. We conclude that $\text{lrr}(\lambda)$ contains at most $|\Phi^+ \setminus \Phi'|$ elements. It is well known that $d = |\Phi^+ \setminus \Phi'|$, which finishes the proof. \square

We can now formulate a simple criterion for an irreducible equivariant vector bundle on an isotropic Grassmannian to be Ulrich.

Lemma 2.4. *An irreducible equivariant vector bundle \mathcal{U}^λ on X is Ulrich if and only if*

$$\text{lrr}(\lambda) = \{1, 2, \dots, d\}.$$

Proof. Given an irreducible equivariant vector bundle \mathcal{U}^λ , one has $\mathcal{U}^\lambda(t) \simeq \mathcal{U}^{\lambda+t\omega_k}$. It follows from the Borel–Bott–Weil theorem that $\text{lrr}(\lambda) = \{1, 2, \dots, d\}$ if and only if $H^\bullet(X, \mathcal{U}^\lambda(t)) = 0$ for $-d \leq t < 0$. Thus, the only if part.

In Lemma 2.3 we have shown that there can be at most d rational intersection points of the line $\lambda + \rho + t\omega_k$ with the walls of the Weyl chambers. We conclude that if $\text{lrr}(\lambda) = \{1, 2, \dots, d\}$, as the maximum number of intersections is reached, the ray $\{\lambda + \rho + t\omega_k\}_{t \geq 0}$ lies in the interior of a single Weyl chamber.

It follows from the Borel–Bott–Weil theorem that an equivariant vector bundle can have at most one nontrivial cohomology group, and \mathcal{U}^μ and \mathcal{U}^ν have nontrivial cohomology in the same degree if $\mu + \rho$ and $\nu + \rho$ lie in the same Weyl chamber. As $H^0(X, \mathcal{U}^\lambda(t)) \neq 0$ for $t \gg 0$, we deduce that $H^0(X, \mathcal{U}^\lambda(t)) \neq 0$ and $H^i(X, \mathcal{U}^\lambda(t)) = 0$ for all $t \geq 0$ and $0 < i \leq d$. A similar argument shows that $H^i(X, \mathcal{U}^\lambda(t)) = 0$ for all $t < -d$ and $0 \leq i < d$. \square

3. SYMPLECTIC GRASSMANNIANS

Let $2 \leq k \leq n$ be integer numbers and let V be a $2n$ -dimensional vector space equipped with a non-degenerate symplectic form.¹ Let $\mathbf{G} = \text{Sp}(V)$ be the corresponding symplectic group. We identify $P_{\mathbf{G}}$ with the set of integral vectors $\mathbb{Z}^n \subset \mathbb{R}^n$. The root system Φ consists of the vectors $\pm 2e_i$ and $\pm(e_i \pm e_j)$ for $i \neq j$, the simple roots are $\varepsilon_i = e_i - e_{i+1}$ for $i = 1, \dots, n-1$ and $\varepsilon_n = 2e_n$, and the fundamental weights are $\omega_i = e_1 + \dots + e_i$ for $i = 1, \dots, n$. In particular,

$$\rho = (n, n-1, \dots, 1).$$

Let $X = \mathbf{G}/\mathbf{P}$, where $\mathbf{P} = \mathbf{P}_k$ is the maximal parabolic subgroup associated with $\Delta \setminus \{\varepsilon_k\}$. The variety X is nothing but the Grassmannian $\text{IGr}(k, V)$ of k -dimensional subspaces in V isotropic with respect to the form. The dimension of X equals $d = k(2n-k) - k(k-1)/2$.

The L -dominant cone $P_{\mathbb{L}}^+$ is identified with $\lambda \in \mathbb{Z}^n$ for which

$$(1) \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k, \quad \lambda_{k+1} \geq \lambda_{k+2} \geq \dots \geq \lambda_n \geq 0.$$

For example, if one denotes by \mathcal{U} the tautological rank k subbundle, and $\lambda = (\lambda_1, \dots, \lambda_k, 0, \dots, 0) \in P_{\mathbb{L}}^+$ is such that $\lambda_k \geq 0$, then $\mathcal{U}^\lambda \simeq \Sigma^\lambda \mathcal{U}^*$, where Σ^λ is the Schur functor corresponding to λ . In particular, $\mathcal{O}_X(1) = \mathcal{U}^{\omega_k} \simeq \Lambda^k \mathcal{U}^*$.

¹The case $k = 1$ is trivial as $X \simeq \mathbb{P}(V)$.

For a given $\lambda \in P_{\mathbb{L}}^+$ define auxiliary sequences $\alpha \in \mathbb{Z}^k$ and $\beta \in \mathbb{Z}^{n-k}$ by

$$(2) \quad (\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{n-k}) = \lambda + \rho.$$

Condition (1) now reads

$$(3) \quad \alpha_1 > \alpha_2 > \dots > \alpha_k, \quad \beta_1 > \beta_2 > \dots > \beta_{n-k} > 0.$$

Proposition 3.1.

$$\text{lrr}(\lambda) = \{\alpha_i \pm \beta_j\}_{1 \leq i \leq k, 1 \leq j \leq n-k} \cup \mathbb{Z} \cap \left\{ \frac{\alpha_i + \alpha_j}{2} \right\}_{1 \leq i \leq j \leq k}$$

Proof. The weight

$$\lambda + \rho - t\omega_k = (\alpha_1 - t, \alpha_2 - t, \dots, \alpha_l - t, \beta_1, \beta_2, \dots, \beta_{n-k})$$

is singular if and only if it is orthogonal to one of the roots of \mathbf{G} . The roots are $\pm 2e_i$ and $\pm(e_i \pm e_j)$ for $i \neq j$. In particular, a weight is singular if and only if (i) one of the terms equals zero, or (ii) two different terms coincide, or (iii) a couple of terms sum up to zero. It follows from (3) that the three options are equivalent to (i) $t = \alpha_i$, (ii) $t = \alpha_i - \beta_j$, and (iii) $t = (\alpha_i + \alpha_j)/2$ or $t = \alpha_i + \beta_j$. \square

Remark 3.2. From Lemma 2.3 we deduce that if \mathcal{U}^λ is Ulrich, then all the numbers $\alpha_i \pm \beta_j$ and $(\alpha_i + \alpha_j)/2$ are distinct and integer.

3.1. General isotropic Grassmannians. The goal of this section is to prove the following statement.

Proposition 3.3. *There are no irreducible equivariant Ulrich bundles on $\mathbf{LGr}(k, V)$ for $2 < k < n$.*

Lemma 3.4. *If there exists an irreducible equivariant Ulrich bundle on X , then $\dim X$ is odd.*

Proof. Assume that $\text{lrr}(\lambda) = \{1, 2, \dots, d\}$. From (3) and Proposition 3.1 we get

$$\min \text{lrr}(\lambda) = \alpha_k - \beta_1 = 1 \quad \text{and} \quad \max \text{lrr}(\lambda) = \alpha_1 + \beta_1 = d.$$

Now, according to Remark 3.2,

$$\mathbb{Z} \supset \text{lrr}(\lambda) \ni \frac{\alpha_1 + \alpha_k}{2} = \frac{(\alpha_1 + \beta_1) + (\alpha_k - \beta_1)}{2} = \frac{d+1}{2}.$$

Thus, d is necessarily odd. \square

Proof of Proposition 3.3. Let us introduce some temporary notation: for integers a and b we will write $a \prec b$ if $b = a + 1$, and similarly $a \succ b$ if $a = b + 1$. Put $l = n - k$.

Assume that λ corresponds to an Ulrich bundle. Then

$$(4) \quad \text{lrr}(\lambda) = \{1, 2, \dots, d\}.$$

Put $D = (d - 1)/2$ and

$$(5) \quad \alpha_{ij}^\pm = \alpha_i \pm \beta_j - \frac{d+1}{2}, \quad \alpha_{ij} = \frac{\alpha_i + \alpha_j}{2} - \frac{d+1}{2}.$$

One immediately observes that

$$(6) \quad \alpha_{ij}^\pm = \alpha_{ii} \pm \beta_j, \quad \alpha_{ij} = \frac{\alpha_{ii} + \alpha_{jj}}{2} \quad \text{for } 1 \leq i < j \leq k.$$

It follows from (4) that the values α_{ij}^\pm and α_{ij} are distinct and fill the integer range $[D, \dots, -D]$. Also, from (3) we get inequalities

$$(7) \quad \begin{aligned} & \alpha_{ij} \leq \alpha_{pq} && \text{if } i \geq p \text{ and } j \geq q, \\ & \alpha_{i1}^+ > \alpha_{i2}^+ > \dots > \alpha_{il}^+ > \alpha_{ii} > \alpha_{il}^- > \dots > \alpha_{i2}^- > \alpha_{i1}^- && \text{for all } 1 \leq i \leq k, \\ & \alpha_{1j}^+ > \alpha_{2j}^+ > \dots > \alpha_{kj}^+, & \alpha_{kj}^- < \dots < \alpha_{2j}^- < \alpha_{1j}^- && \text{for all } 1 \leq j \leq l. \end{aligned}$$

From the proof of Lemma 3.4 and (5) we see that

$$(8) \quad \alpha_{11}^+ = D, \quad \alpha_{k1}^- = -D, \quad \alpha_{11} = -\alpha_{kk}, \quad \alpha_{1k} = 0.$$

Let us first assume that $(\beta_1, \beta_2, \dots, \beta_l) \neq (l, l-1, \dots, 1)$. The last condition is equivalent to the existence of such a $t \in \{1, \dots, l\}$ that

$$\beta_1 \succ \beta_2 \succ \dots \succ \beta_t > 1.$$

It follows from (6) that

$$D = \alpha_{11}^+ \succ \alpha_{12}^+ \succ \dots \succ \alpha_{1t}^+ \not\succ \alpha_{11}.$$

From inequalities 7 we deduce that the next position after α_{1t}^+ must be taken by α_{21}^+ . In particular,

$$(9) \quad \alpha_{21}^+ = \alpha_{22} + \beta_1 = \alpha_{11}^+ - t.$$

A similar argument shows that

$$-D = \alpha_{k1}^- \prec \alpha_{k2}^- \prec \dots \prec \alpha_{kt}^- \not\prec \alpha_{k-1,1}^-,$$

which implies

$$(10) \quad \alpha_{k-1,1}^- = \alpha_{k-1,k-1} - \beta_1 = \alpha_{11}^- + t.$$

Combining (9) and 10, we deduce

$$\alpha_{2,k-1} = \frac{\alpha_{22} + \alpha_{k-1,k-1}}{2} = \frac{\alpha_{11} + \alpha_{kk}}{2} = \alpha_{1k}.$$

That contradicts the assumption that all the values α_{ij} are distinct.

Finally, in case $(\beta_1, \beta_2, \dots, \beta_l) = (l, l-1, \dots, 1)$ one has

$$D = \alpha_{11}^+ \succ \alpha_{12}^+ \succ \dots \succ \alpha_{1k}^+ \succ \alpha_{11} \succ \alpha_{1k}^- \succ \dots \succ \alpha_{12}^- \succ \alpha_{11}^- > \alpha_{12} > \alpha_{21}^+ \succ \alpha_{22}^+ \succ \dots \succ \alpha_{2k}^+ \succ \alpha_{22}.$$

Inequalities (7) imply that α_{12} is the only possible immediate predecessor of α_{11}^- , thus

$$\alpha_{12} = \frac{\alpha_{11} + \alpha_{22}}{2} = \alpha_{11} - (l+1) \quad \Rightarrow \quad \alpha_{22} = \alpha_{11} - 2(l+1).$$

A similar argument shows that

$$\alpha_{k-1,k} = \frac{\alpha_{k-1,k-1} + \alpha_{kk}}{2} = \alpha_{kk} + (l+1) \quad \Rightarrow \quad \alpha_{k-1,k-1} = \alpha_{kk} + 2(l+1).$$

Combining the last two equalities we deduce that

$$\alpha_{2,k-1} = \frac{\alpha_{22} + \alpha_{k-1,k-1}}{2} = \frac{\alpha_{11} + \alpha_{kk}}{2} = \alpha_{11},$$

which once again contradicts the assumption that all α_{ij} are distinct. \square

3.2. Isotropic Grassmannians of planes. We would like to classify all irreducible equivariant Ulrich bundles on isotropic Grassmannians of planes. Thus, throughout this section $k = 2$ and $X = \text{IGr}(2, V)$.

Proposition 3.5. *Let p be a positive divisor of $n-1$ such that $(n-1)/p = 2q+1$ is odd. Then the bundle on $\text{IGr}(2, V)$ corresponding to the weight*

$$(11) \quad \lambda = (n-2+p, n-1-p, \underbrace{2pq, \dots, 2pq}_{2p}, \underbrace{2p(q-1), \dots, 2p(q-1)}_{2p}, \dots, \underbrace{2p, \dots, 2p}_{2p}, \underbrace{0, \dots, 0}_{p-1})$$

is Ulrich. Moreover, all irreducible equivariant Ulrich bundles on $\text{IGr}(2, V)$ are of this form.

Proof. We will freely use the notation introduced along the proof of Proposition 3.3. Remark that the only α_{ij} we are left with are $\alpha_{12} = 0$ and $\alpha_{11} = -\alpha_{22}$. Let us also spell out $d = 4n - 5$, $l = n - 2$, and $D = 2n - 3$.

Assume that the values α_{ij}^\pm and α_{ij} cover the range $[D, \dots, -D]$, and put $p = \alpha_{11}$. Let us show by induction on l that $l = 2pq + (p - 1)$ for some integer $q \geq 0$ and that

$$(12) \quad \begin{array}{ccccccc} \beta_1 = 2n - 3 - p & \beta_{2p+1} = 2n - 3 - 5p & \dots & \beta_{2p(q-1)+1} = 5p - 1 & \beta_{2pq+1} = p - 1 \\ \beta_2 = 2n - 4 - p & \beta_{2p+2} = 2n - 4 - 5p & \dots & \beta_{2p(q-1)+2} = 5p - 2 & \vdots \\ \vdots & \vdots & & \vdots & \beta_l = 1 \\ \beta_{2p} = 2n - 2 - 3p & \beta_{4p} = 2n - 2 - 7p & \dots & \beta_{2p(q-1)+2p} = 3p \end{array}$$

As $l = n - 2$, the equality $l = 2pq + (p - 1)$ is equivalent to $n - 1 = p(2q + 1)$, which is what we would like to show. Also, by (2) and (5) one has

$$\lambda_1 = \alpha_{11} + \frac{d+1}{2} - n, \quad \lambda_2 = \alpha_{22} + \frac{d+1}{2} - (n-1), \quad \lambda_i = \beta_{i-2} - (n+1-i) \quad \text{for } 2 < i \leq n.$$

Simplifying these formulas, one comes to a conclusion that (12) corresponds exactly to (11). Thus, we are indeed about to prove an equivalent statement.

In the base case $l = 3$ we need to find all possible integers $p, \beta_1 > 0$ such that $\pm p \pm \beta_1$ and $\pm p$ are distinct and take all the values $\{\pm 1, \pm 2, \pm 3\}$. One immediately checks that the only possibility is $p = 2$ and $\beta_1 = 1$.

Let us prove the inductive step. Recall that the $\alpha_{11}^+ = D$ is necessarily the maximal element. From (7) we know that the only elements that can take the values $\{\pm 1, \pm 2, \dots, \pm(p-1)\}$ are α_{ij}^\pm . Thus, by the pigeonhole principle $l \geq p - 1$. Next, if $l = p - 1$, then by the same principle $\alpha_{21}^- < p$. As a consequence,

$$D = \alpha_{11}^+ \succ \alpha_{12}^+ \succ \dots \succ \alpha_{1,p-1}^+ \succ \alpha_{11} = p,$$

which implies

$$\beta_1 \succ \beta_2 \succ \dots \succ \beta_{p-1} = 1.$$

That is exactly what (12) gets reduced to.

Finally, assume that $l > p - 1$. By the pigeonhole principle again and the order on $\alpha_{2,i}^-$ we argue that $\alpha_{21}^+ = \alpha_{11}^+ - 2p > \alpha_{11}$. Looking at (7), we deduce that

$$D = \alpha_{11}^+ \succ \alpha_{12}^+ \succ \dots \succ \alpha_{1,2p}^+ \succ \alpha_{21}^+.$$

In particular,

$$\beta_1 \succ \beta_2 \succ \dots \succ \beta_{2p} \not\succ \beta_{2p+1},$$

which implies

$$(13) \quad \begin{aligned} D &= \alpha_{11}^+ \succ \alpha_{12}^+ \succ \dots \succ \alpha_{1,2p}^+ \succ \alpha_{21}^+ \succ \alpha_{22}^+ \succ \dots \succ \alpha_{2,2p}^+, \\ \alpha_{1,2p}^- &\succ \dots \succ \alpha_{12}^- \succ \alpha_{11}^- \succ \alpha_{2,2p}^- \succ \dots \succ \alpha_{22}^- \succ \alpha_{21}^- = -D. \end{aligned}$$

We conclude that

$$(14) \quad (\beta_1, \beta_2, \dots, \beta_{2p}) = (D - p, D - p - 1, \dots, D - 3p + 1).$$

It follows from (13) that the range $[D - 4p, \dots, -D + 4p]$ is covered by the values

$$\alpha_{11} = p, \quad \alpha_{12} = 0, \quad \alpha_{22} = -p, \quad \text{and} \quad \left\{ \alpha_{ij}^\pm \right\}_{2p < j \leq l}.$$

Applying the induction hypothesis for $l' = l - 2p$ we find that $l - 2p = 2pq' + (p - 1)$. In particular, $l = 2pq + (p - 1)$ for $q = q' + 1$. We also get the exact values of β_j for $j > 2p$ in terms of p and $n' = n - 2p$. Rewriting these values back in terms of p and n together with (14) proves (12). \square

Remark that for any n one has an obvious choice $p = n - 1$. Then $(n - 1)/p = 1$ is odd and the corresponding weight is $(2n - 3, 0, \dots, 0)$. Moreover, if $n - 1$ has no odd divisors apart from 1, this choice of p is the only possible. Summarising, we get the following statement.

Corollary 3.6. *For any n the bundle $S^{2n-3}\mathcal{U}^*$ on $\mathrm{IGr}(2, 2n)$ is Ulrich. If $n = 2^r + 1$, it is the only irreducible equivariant Ulrich bundle up to isomorphism.*

Example 3.7. The following are the smallest isotropic Grassmannians of planes that admit two and three nonisomorphic irreducible Ulrich bundles respectively.

- (1) There are exactly two nonisomorphic equivariant irreducible Ulrich bundles on $\mathrm{IGr}(2, 8)$:

$$S^5\mathcal{U}^* \quad \text{and} \quad \mathcal{U}^*(2) \otimes \Sigma^{[2,2]}(\mathcal{U}^\perp/\mathcal{U}).$$

- (2) There are exactly three nonisomorphic equivariant irreducible Ulrich bundles on $\mathrm{IGr}(2, 20)$:

$$S^{17}\mathcal{U}^*, \quad S^5\mathcal{U}^*(6) \otimes \Sigma^{[6,6,6,6,6]}(\mathcal{U}^\perp/\mathcal{U}), \quad \text{and} \quad \mathcal{U}^*(8) \otimes \Sigma^{[8,8,6,6,4,4,2,2]}(\mathcal{U}^\perp/\mathcal{U}).$$

Here $\Sigma^{[\mu]}$ denotes the symplectic Schur functor corresponding to μ .

4. ODD ORTHOGONAL GRASSMANNIANS

We would like to deal now with type B_n . Namely, let $\mathbf{G} = \mathrm{SO}(V)$, where V is a $(2n + 1)$ -dimensional vector space equipped with a non-degenerate symmetric form. We identify the root system of \mathbf{G} with the set of all integer vectors of length $\sqrt{2}$ or 1. For the simple roots we choose $\varepsilon_i = e_i - e_{i+1}$ for $i = 1, \dots, n - 1$, and $\varepsilon_n = e_n$. The fundamental weights then are $\omega_i = e_1 + \dots + e_i$ for $i = 1, \dots, n - 1$ and $\omega_n = \frac{1}{2}(e_1 + \dots + e_n)$, while

$$\rho = \left(n - \frac{1}{2}, n - \frac{3}{2}, \dots, \frac{1}{2}\right).$$

We would like to classify equivariant irreducible vector bundles on $X = \mathbf{G}/\mathbf{P}_k \simeq \mathrm{OGr}(k, V)$ for all $k = 1, \dots, n - 1$. The dimension of X is $d = k(2n - k) - k(k + 1)/2$. The weight lattice $P_{\mathbf{G}}$ is identified with the set of vectors with all integer or halfinteger coordinates, $P_{\mathbf{G}} = \mathbb{Z}^n \cup (\frac{1}{2} + \mathbb{Z})^n$, while the dominant cone $P_{\mathbf{L}}^+$ consists of all $\lambda \in P_{\mathbf{G}}$ such that

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k, \quad \lambda_{k+1} \geq \dots \geq \lambda_n \geq 0.$$

Let us again introduce the notation

$$(\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_{n-k}) = \lambda + \rho$$

For $\lambda \in P_{\mathbf{L}}^+$ we deduce inequalities

$$(15) \quad \alpha_1 > \alpha_2 > \dots > \alpha_k, \quad \beta_1 > \beta_2 > \dots > \beta_{n-k} > 0.$$

Remark that the only difference between (15) and (3) is that in type B_n the terms α_i and β_j are allowed to be halfinteger all at once.

The root systems in type B_n and C_n are extremely similar. An argument analagous to the one used in Proposition 3.1 shows that

$$\mathrm{lrr}(\lambda) = \{\alpha_i \pm \beta_j\}_{1 \leq i \leq k, 1 \leq j \leq n-k} \cup \mathbb{Z} \cap \left\{ \frac{\alpha_i + \alpha_j}{2} \right\}_{1 \leq i \leq j \leq k}.$$

As the cardinality of $\mathrm{lrr}(\lambda)$ is at most d , we deduce that all the values $(\alpha_i + \alpha_j)/2$ are integer. In particular, all α_i are integer, thus β_j are forced to be as well. We come to the following conclusion.

Proposition 4.1. *A weight $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ corresponds to an Ulrich bundle on $\mathrm{OGr}(k, 2n+1)$ if and only if $(\lambda_1 - \frac{1}{2}, \lambda_2 - \frac{1}{2}, \dots, \lambda_n - \frac{1}{2})$ is a weight corresponding to an Ulrich bundle for $\mathrm{IGr}(k, 2n)$.*

Proof. The conditions on α and β under which the bundle \mathcal{U}^λ is Ulrich are exactly the same in types C_n and B_n . Thus, λ is Ulrich for $\mathrm{OGr}(k, 2n+1)$ if and only if $\lambda + \rho_{B_n} - \rho_{C_n}$ is Ulrich for $\mathrm{IGr}(k, 2n)$. \square

Using Proposition 3.5 we can now find all equivariant Ulrich bundles on $\mathrm{OGr}(2, 2n+1)$.

Corollary 4.2. *Every irreducible equivariant Ulrich bundle \mathcal{U}^λ on $\mathrm{OGr}(2, 2n+1)$ corresponds to λ of the form*

$$(16) \quad \lambda = (n - \frac{3}{2} + p, n - \frac{1}{2} - p, \underbrace{2pq + \frac{1}{2}, \dots, 2pq + \frac{1}{2}}_{2p}, \dots, \underbrace{2p + \frac{1}{2}, \dots, 2p + \frac{1}{2}}_{2p}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{p-1}),$$

where p is a positive divisor of $n-1$ such that $(n-1)/p = 2q+1$ is odd.

Let \mathcal{S} denote the spinor bundle on X , namely,

$$\mathcal{S} = \mathcal{U}^{(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})}.$$

Corollary 4.3. *Up to isomorphism \mathcal{S} is the only irreducible equivariant Ulrich bundle on an odd-dimensional quadric Q^{2n-1} .*

Proof. The only irreducible Ulrich bundle on \mathbb{P}^{2n-1} is \mathcal{O} , which corresponds to $\lambda = (0, 0, \dots, 0)$. \square

Corollary 4.4. *The only nonmaximal odd orthogonal Grassmannians that admit an irreducible equivariant Ulrich bundle are the odd-dimensional quadrics $\mathrm{OGr}(1, V) \simeq Q^{2n-1}$ and the Grassmannians of isotropic planes $\mathrm{OGr}(2, V)$.*

5. EVEN ORTHOGONAL GRASSMANNIANS

Let us now deal with groups of type D_n . Namely, let $\mathbf{G} = \mathrm{SO}(V)$, where V is a $2n$ -dimensional vector space equipped with a non-degenerate symmetric form. We identify the root of \mathbf{G} system with the set of vectors $\pm(e_i \pm e_j)$, $i \neq j$. The simple roots are $\varepsilon_i = e_i - e_{i+1}$ for $i = 1, \dots, n-1$ and $\varepsilon_n = e_{n-1} + e_n$, while the fundamental weights are $\omega_i = e_1 + \dots + e_i$ for $i = 1, \dots, n-2$, and

$$\begin{aligned} \omega_{n-1} &= \frac{1}{2}(e_1 + \dots + e_{n-1} - e_n), \\ \omega_n &= \frac{1}{2}(e_1 + \dots + e_{n-1} + e_n). \end{aligned}$$

Thus, $\rho = (n-1, n-2, \dots, 0)$.

Let $X = \mathbf{G}/\mathbf{P}_k$. The dimension of X is $d = k(2n-k) - k(k+1)/2$. It is well known that $X \simeq \mathrm{OGr}(k, V)$ for $k = 1, \dots, n-2$, while $\mathbf{G}/\mathbf{P}_{n-1}$ and \mathbf{G}/\mathbf{P}_n are isomorphic to the two connected components of the Grassmannian of maximal isotropic subspaces $\mathrm{OGr}(n, V)$. The latter two varieties will be treated in Section 6, here we assume $1 \leq k \leq n-2$.

As in type B_n , the weight lattice $P_{\mathbf{G}}$ is identified with the set of all (half)integer vectors $\mathbb{Z}^n \cup (\frac{1}{2} + \mathbb{Z})^n$, while the dominant cone $P_{\mathbf{L}}^+$ consists of all $\lambda \in P_{\mathbf{G}}$ such that

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k, \quad \lambda_{k+1} \geq \dots \geq |\lambda_n|.$$

We once again introduce the notation

$$(\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_{n-k}) = \lambda + \rho$$

For $\lambda \in P_{\mathbf{L}}^+$ one has inequalities

$$(17) \quad \alpha_1 > \alpha_2 > \dots > \alpha_k, \quad \beta_1 > \beta_2 > \dots > |\beta_{n-k}|.$$

An argument similar to the one used Proposition 3.1 shows that

$$(18) \quad \text{Irr}(\lambda) = \{\alpha_i \pm \beta_j\}_{1 \leq i \leq k, 1 \leq j \leq n-k} \cup \mathbb{Z} \cap \left\{ \frac{\alpha_i + \alpha_j}{2} \right\}_{1 \leq i < j \leq k}.$$

Remark 5.1. Equality (18) together with $\beta_{n-k} = \lambda_n$ imply that λ corresponds to an Ulrich bundle if and only if the weight $\lambda' = (\lambda_1, \lambda_2, \dots, \lambda_{n-1}, -\lambda_n)$ does. Moreover, \mathcal{U}^λ can not be Ulrich as long as $\lambda_n = 0$. Thus, it is sufficient to classify irreducible equivariant Ulrich bundles with $\lambda_n > 0$.

5.1. General orthogonal Grassmannians. We are first going to prove the following statement.

Proposition 5.2. *There are no irreducible equivariant Ulrich bundles on $\text{OGr}(k, V)$ for $3 < k \leq n - 2$.*

Proof. The proof is very similar to the proof of Proposition 3.3. Put $l = n - k$ and write $a \prec b$ ($a \succ b$) whenever $a + 1 = b$ ($a = b + 1$ respectively). Let us assume that $\text{Irr}(\lambda) = \{1, \dots, d\}$. Then necessarily

$$\alpha_1 + \beta_1 = d, \quad \alpha_k - \beta_1 = 1, \quad \alpha_1 + \alpha_k = d + 1.$$

To keep notation consistent, we introduce $\alpha_{ij}^\pm = \alpha_i \pm \beta_j$ and $\alpha_{ij} = \frac{\alpha_i + \alpha_j}{2} - \frac{d+1}{2}$. According to our assumption, the values α_{ij}^\pm together with α_{ij} for $1 \leq i < j \leq k$ are all integer, distinct and span the range $[D, \dots, -D]$, where $D = (d - 1)/2$. According to Remark 5.1, we can also restrict ourselves to the case $\beta_l > 0$, thus inequalities (7) remain valid.

In case $\beta = (l - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2})$ one has

$$\alpha_{i1}^+ \succ \alpha_{i2}^+ \succ \dots \succ \alpha_{il}^+ \succ \alpha_{il}^- \succ \dots \succ \alpha_{i2}^- \succ \alpha_{i1}^-$$

for all $i = 1, \dots, k$. In particular, $\alpha_{11}^- \succ \alpha_{12}$ and $\alpha_{k,1}^+ \prec \alpha_{k-1,k}$, which implies that $\alpha_{2,k-1} = \alpha_{1,k}$.

In case $\beta \neq (l - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2})$ we claim that

$$(19) \quad \alpha_{11}^+ \succ \alpha_{12}^+ \succ \dots \succ \alpha_{1t}^+ \succ \alpha_{21}^+$$

for some $1 \leq p \leq k$. Indeed, otherwise

$$\alpha_{11}^+ \succ \alpha_{12}^+ \succ \dots \succ \alpha_{1l}^+ \succ \alpha_{12}.$$

In particular, $\alpha_1 + \beta_1 = \frac{\alpha_1 + \alpha_2}{2} + 1$. As $\beta_1 \geq \frac{1}{2}$ and $\alpha_2 \leq \alpha_1 - 1$, we deduce that $\beta = (l - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2})$. Similarly, we argue that

$$\alpha_{k,1}^- \prec \alpha_{k,2}^- \prec \dots \prec \alpha_{k,t}^- \prec \alpha_{k-1,1}^-$$

for the same p as in (19), which implies that

$$\alpha_2 = \alpha_1 - p, \quad \text{and} \quad \alpha_{k-1} = \alpha_k + p.$$

Thus, $\alpha_{2,k-1} = \alpha_{1,k}$, which contradicts the assumption that all the values are distinct. \square

5.2. Small even orthogonal Grassmannians. Let us classify irreducible equivariant Ulrich bundles on $\text{OGr}(k, 2n)$ for $k \leq 3$. We will freely use the notation introduced along the proof of Proposition 5.2.

5.2.1. Even-dimensional quadrics.

Proposition 5.3. *The only irreducible equivariant Ulrich bundles on an even-dimensional quadric Q^{2n-2} up to isomorphism are the spinor bundles \mathcal{S}^\pm , which correspond to the weights $\lambda^\pm = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \pm \frac{1}{2})$.*

Proof. According to inequalities (7), the only possible complete ordering is

$$2n - 2 = \alpha_1 + \beta_1 \succ \dots \succ \alpha_1 + |\beta_{n-1}| \succ \alpha_1 - |\beta_{n-1}| \succ \dots \succ \alpha_1 - \beta_1 = 1.$$

Thus, $(\alpha, \beta_1, \dots, \beta_{n-2}, |\beta_{n-1}|) = (n - \frac{1}{2}, n - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2})$. It remains to subtract $\rho = (n - 1, \dots, 1, 0)$. \square

5.2.2. Even orthogonal Grassmannians of planes.

Proposition 5.4. *Let $p \in \frac{1}{2} + \mathbb{Z}$ and $q \in \mathbb{Z}$ be nonnegative integers such that $n - 2 = p(2q + 1) - \frac{1}{2}$. Then the bundle on $X = \text{OGr}(2, 2n)$ corresponding to the weight*

$$(20) \quad \lambda = (n - 2 + p, n - 1 - p, \underbrace{2pq + \frac{1}{2}, \dots, 2pq + \frac{1}{2}}_{2p}, \underbrace{2p(q - 1) + \frac{1}{2}, \dots, 2p(q - 1) + \frac{1}{2}}_{2p}, \dots, \underbrace{2p + \frac{1}{2}, \dots, 2p + \frac{1}{2}}_{2p}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{p - \frac{1}{2}})$$

is Ulrich. Moreover, all irreducible equivariant Ulrich bundles on X are of this form up to negating λ_n .

Proof. The situation is very similar to that in type C_n . According to Remark 5.1, we may assume that $\lambda_n > 0$. The dimension of X equals $d = 4n - 7$, thus $D = 2n - 4$. Let $\alpha_{1,1} = -\alpha_{2,2} = p$. It is enough to show that if the values $\alpha_{i,j}^\pm$ and $\alpha_{1,2} = 0$ cover the whole integer range $[2n + 4, \dots, -2n + 4]$, then

$$(21) \quad \begin{array}{ccccccc} \beta_1 = 2n - 4 - p & \beta_{2p+1} = 2n - 4 - 5p & \dots & \beta_{2p(q-1)+1} = 5p - 1 & \beta_{2pq+1} = p - 1 & & \\ \beta_2 = 2n - 5 - p & \beta_{2p+2} = 2n - 5 - 5p & \dots & \beta_{2p(q-1)+2} = 5p - 2 & & \vdots & \\ \vdots & \vdots & & \vdots & & \beta_{n-2} = \frac{1}{2} & \\ \beta_{2p} = 2n - 3 - 3p & \beta_{4p} = 2n - 3 - 7p & \dots & \beta_{2p(q-1)+2p} = 3p & & & \end{array}$$

As in the proof of Proposition 3.5, we show the latter by induction on l . For the base case we take $\alpha_{2,1}^+ = -p + \beta_1 < p$. Then, by pigeonhole principle $2l = 2p - 1$ and $\beta = (p - 1, \dots, \frac{3}{2}, \frac{1}{2})$. Otherwise,

$$\alpha_{1,1}^+ \succ \alpha_{1,2}^+ \succ \dots \succ \alpha_{1,t}^+ \succ \alpha_{2,t}^+$$

for some $1 \leq t \leq l$, from which we deduce that $t = 2p$ and

$$D - p = \beta_1 \succ \beta_2 \succ \dots \succ \beta_{2p},$$

and reduce the statement to $l' = l - 2p$. □

Example 5.5. For any even orthogonal Grassmannian of planes $\text{OGr}(2, 2n)$ there are at least four non-isomorphic irreducible equivariant Ulrich vector bundles. In terms of Proposition 5.4 they correspond to the values $p = n - \frac{3}{2}$, $q = 0$, and $p = \frac{1}{2}$, $q = n - 2$ respectively. The former values produce the bundles $S^{2n-4}\mathcal{U}^* \otimes \mathcal{S}^\pm$, where $S^{2n-4}\mathcal{U}^*$ is the $(2n - 4)$ -th symmetric power of the dual tautological rank 2 bundle, and \mathcal{S}^\pm are the spinor bundles. In the latter case

$$\lambda = (n - \frac{3}{2}, n - \frac{3}{2}, n - \frac{3}{2}, n - \frac{5}{2}, \dots, \frac{5}{2}, \pm \frac{3}{2}).$$

5.2.3. Even orthogonal Grassmannians of 3-spaces.

We finally turn to the case $k = 3$.

Proposition 5.6. *The Grassmannian $\text{OGr}(3, 2n)$ carries an irreducible equivariant Ulrich bundle if and only if $n - 3 = 2q$. In the latter case the only equivariant irreducible Ulrich bundles are isomorphic to $\mathcal{U}^{\lambda^\pm}$ for*

$$(22) \quad \lambda^\pm = (2n - 4, 2n - 5, 2n - 6, 2n - 6, 2n - 6, 2n - 10, 2n - 10, \dots, 4, \pm 4).$$

Proof. According to Remark 5.1, we may assume that $\lambda_n = \beta_l > 0$. Let \mathcal{U}^λ be Ulrich. Following the proof of Proposition 5.2 we see that $\alpha_{1,1} = -\alpha_{3,3}$ and $\alpha_{1,2} = \alpha_{2,3}$, which implies $\alpha_{2,2} = 0$. Denote $p = \alpha_{1,1}$. Remark that $p \in 2\mathbb{Z}$ as $\alpha_{1,2} = p/2 \in \mathbb{Z}$. Let us show that $\beta_1 \geq p$. Indeed, otherwise from inequalities

$$p > \beta_1 > \beta_2 > \dots > \beta_l > 0$$

we deduce from (7) that none of the α_{ij}^\pm can be equal to $p \in [D, \dots, -D]$. Now, we argue by induction that $l = 2q$ is even and

$$(23) \quad \beta = (D - 2, D - 3, D - 6, D - 7, \dots, 5, 4), \quad p = 2.$$

The base cases are $l = 0$ and $l = 1$. In the former p trivially equals 2, while in the latter one can show with brute force that λ does not exist.

Now, for $l > 1$ we have $\alpha_{2,1}^+ \geq p$, and from (7) one has

$$\alpha_{1,1}^+ \succ \alpha_{1,2}^+ \succ \dots \succ \alpha_{1,t}^+ \succ \alpha_{2,1}^+$$

for some $1 \leq t \leq l$. We immediately conclude that $t = p$ and

$$D - p = \beta_1 \succ \beta_2 \succ \dots \succ \beta_p.$$

It remains to observe that $\alpha_{3,t}^+ = \alpha_{3,1}^+ - 2p$, thus

$$\alpha_{1,1}^+ \succ \dots \succ \alpha_{1,p}^+ \succ \alpha_{2,1}^+ \succ \dots \succ \alpha_{2,p}^+ \succ \alpha_{3,1}^+ \succ \dots \succ \alpha_{3,p}^+,$$

from which we see that by dropping $(\beta_1, \dots, \beta_p)$ the problem gets reduced to $l' = l - p$, and that (23) is equivalent to (22), as $D = 3n - 8$. \square

6. MAXIMAL ISOTROPIC GRASSMANNIANS

6.1. Lagrangian Grassmannians. Let V be a $2n$ -dimensional symplectic vector space and let $X = \text{LGr}(n, V)$ be the Lagrangian Grassmannian of maximal isotropic subspaces. Our goal is to prove the following.

Proposition 6.1. *The only Lagrangian Grassmannian possessing an irreducible equivariant Ulrich bundle is $\text{LGr}(2, 4)$. The only irreducible equivariant Ulrich bundle on $\text{LGr}(2, 4)$ up to isomorphism is \mathcal{U}^* .*

In the Lagrangian case Proposition 3.1 simplifies to

$$\text{lrr}(\lambda) = \left\{ \frac{\alpha_i + \alpha_j}{2} \right\}_{1 \leq i \leq j \leq n}.$$

Recall that \mathcal{U}^λ is Ulrich if and only if $\text{lrr}(\lambda) = \{1, 2, \dots, d\}$. Informally speaking, what we need is to find all integer sequences of length n such that the half sums fill in all the blanks between the elements without any collisions.

Consider a partially ordered set $A = \{a_{ij}\}_{1 \leq i \leq j}$ with elements, indexed by unordered pairs of positive integers, subject to the partial order

$$(24) \quad a_{ij} \leq a_{pq} \quad \text{if } i \leq p \text{ and } j \leq q.$$

Denote by A_n the subset

$$(25) \quad A_n = \{a_{ij}\}_{1 \leq i \leq j \leq n} \subset A.$$

In particular, there is an increasing chain of inclusions $\emptyset = A_0 \subset A_1 \subset \dots \subset A_n \subset \dots \subset A$.

Lemma 6.2. *The partially ordered set $A \setminus A_n$ has a least element*

$$\min(A \setminus A_n) = a_{1,n+1}.$$

Proof. Immediately follows from (24). \square

Proposition 6.3. *There exists a unique order-preserving mapping $T : A \rightarrow \mathbb{Z}_+$ from A to the set of positive integers \mathbb{Z}_+ such that*

$$(26) \quad T(\min(A \setminus A_n)) = \min(\mathbb{Z}_+ \setminus T(A_n)) \quad \text{for all } n > 0,$$

$$(27) \quad T(a_{ij}) = \frac{T(a_{ii}) + T(a_{jj})}{2} \quad \text{for all } 1 \leq i < j.$$

If we denote $\alpha_{ij} = T(a_{ij})$, then $\alpha_{11} = 1$, $\alpha_{ii} = 4i - 5$ for all $i > 1$.

Proof. We start with an observation that condition (27) is consistent with the partial order given by (24). Also, the equality $\alpha_{11} = 1$ obviously follows from (26) and $A_0 = \emptyset$. Let us prove by induction that for all $n > 1$

$$(28) \quad \alpha_{nn} = 4n - 5, \quad \{1, 2, \dots, 2n - 1\} \subseteq T(A_n), \quad 2n \notin T(A_n).$$

The base case, $n = 2$, easily follows from (26). Condition (27) implies $\alpha_{11} = 1$, $\alpha_{12} = 2$, $\alpha_{22} = 3$.

Assume that (28) holds for all $l = 2, \dots, n$. As $2n \notin T(A_n)$ and all the smaller positive integers are contained in $T(A_n)$ by inductive hypothesis, condition (26) dictates $\alpha_{1,n+1} = T(\min(A \setminus A_n)) = 2n$. From (27) we deduce that $\alpha_{n+1,n+1} = 2\alpha_{1,n+1} - \alpha_{11} = 4n - 1$. Moreover, as $\alpha_{1,n+1} = 2n$ and $\alpha_{2,n+1} = (\alpha_{22} + \alpha_{n+1,n+1})/2 = 2n + 1$, we conclude that $\{1, 2, \dots, 2n + 1\} \subseteq T(A_{n+1})$. It remains to show that $2n + 2 \notin T(A_{n+1})$. Remark that the only even elements in $T(A_{n+1})$ are α_{1i} for $i = 2, \dots, n + 1$. Indeed, the first condition in (28) implies

$$\alpha_{ij} = \frac{\alpha_{ii} + \alpha_{jj}}{2} = 2(i + j) - 5 \quad \text{for all } 1 < i \leq j \leq n + 1.$$

On the other hand, $\alpha_{1j} = (\alpha_{11} + \alpha_{jj})/2 = 2j - 2$ for all $j = 2, \dots, n + 1$. In particular, $2n + 2$ is not contained in $T(A_{n+1})$. \square

We fix the values α_{ij} from Proposition 6.3. The following statement is immediate.

Lemma 6.4. *If the bundle \mathcal{U}^λ is Ulrich, then $\alpha_{n+1-i} = \alpha_{ii}$ for $i = 1, \dots, n$.*

Proof of Proposition 6.1. As α_1 is the largest element in $\text{lrr}(\lambda + \rho)$, one of the necessary conditions for \mathcal{U}^λ to be Ulrich is $\alpha_1 = \dim X = n(n + 1)/2$. On the other hand, Lemma 6.4 together with Proposition 6.3 impose $\alpha_1 = \alpha_{nn} = 4n - 5$. Having solved the quadratic equation, one is left with two possible cases: $n = 2$ or $n = 5$. The former corresponds to $\alpha_1 = 3$ and $\alpha_2 = 1$, which in turn is equivalent to $\lambda = (1, 0)$. In the latter case $\alpha_3 = (\alpha_2 + \alpha_4)/2$, thus $\text{lrr}(\lambda)$ can not be of cardinality d . \square

6.1.1. Maximal orthogonal Grassmannians. We are left with the case $X = \mathbf{G}/\mathbf{P}_n$, where \mathbf{G} is of type B_n or D_n . According to our convention, in both cases the lattice $P_{\mathbf{G}}$ is identified with $\mathbb{Z}^n \cup (\frac{1}{2} + \mathbb{Z})^n$, while the \mathbf{L} -dominant weights precisely are those $\lambda = a_1\omega_1 + \dots + a_n\omega_n$ for which $a_i \geq 0$ for $i = 1, \dots, n - 1$ and a_n is arbitrary. It is easy to see that $\lambda \in P^{\mathbf{L}}$ if and only if

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n.$$

Now, the difference with the nonmaximal case comes from the fact that

$$\omega_n = \left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right).$$

If one puts $\alpha = \lambda + \rho$, then

$$\text{lrr}(\lambda) = \begin{cases} \{\alpha_i + \alpha_j\}_{1 \leq i \leq j \leq n}, & \text{in type } B_n, \\ \{\alpha_i + \alpha_j\}_{1 \leq i < j \leq n}, & \text{in type } D_n. \end{cases}$$

Proposition 6.5. *In type B_n the only maximal isotropic Grassmannian carrying an irreducible Ulrich bundle is $\text{OGr}(2, 5)$. The corresponding bundle is trivial.*

Proof. Assume that \mathcal{U}^λ is Ulrich. In particular, $\text{lrr}(\lambda) = \{1, \dots, d\}$. We claim that \mathcal{U}^μ is Ulrich on $\text{LGr}(n, 2n)$ for $\mu = 2\lambda + 2\rho_{B_n} - \rho_{C_n}$. Indeed, $\alpha' = \mu + \rho_{C_n} = 2\alpha$ and $\dim \text{LGr}(n, 2n) = \dim \text{OGr}(n, 2n+1)$, thus

$$\text{lrr}_{C_n}(\mu) = \left\{ \frac{\alpha'_i + \alpha'_j}{2} \right\}_{1 \leq i < j \leq n} = \{\alpha_i + \alpha_j\}_{1 \leq i < j \leq n} = \text{lrr}_{B_n}(\lambda) = \{1, \dots, d\}.$$

It follows from Proposition 6.1 that $n = 2$ and $\mu = 2\lambda + (1, 0) = (1, 0)$, from which $\lambda = 0$. \square

Proposition 6.6. *Every irreducible equivariant Ulrich bundle on $\text{OGr}(2, 4)$ and $\text{OGr}(3, 6)$ is trivial. The dual tautological bundle \mathcal{U}^* is the only irreducible equivariant Ulrich bundle on $\text{OGr}(4, 8)$. There are no irreducible equivariant vector bundles on $\text{OGr}(n, 2n)$ for $n \geq 4$.*

Proof. We are basically interested in those $\alpha \in \mathbb{Z}^n \cup (\frac{1}{2} + \mathbb{Z})^n$ for which $\alpha_1 > \alpha_2 > \dots > \alpha_n$, and

$$\{\alpha_i + \alpha_j\}_{1 \leq i < j \leq n} = \left\{ 1, \dots, \frac{n(n-1)}{2} \right\}.$$

An argument similar to the one used in type C_n shows that for such an α one necessarily has

$$\alpha_n = 0, \alpha_{n-1} = 1, \alpha_{n-2} = 2, \alpha_{n-3} = 4, \alpha_{n-4} = 7, \alpha_{n-5} = 10.$$

Then $\alpha_{n-3} + \alpha_{n-4} = \alpha_{n-1} + \alpha_{n-5}$, so $n \leq 5$. On the other hand, for $n = 5$ the set $\text{lrr} = \{1, \dots, 9, 11\}$ does not contain d . It remains to check (say, by hand) that $\lambda = 0$ for $n = 2, 3$ and $\lambda = (1, 0, 0, 0)$ for $n = 4$ indeed provide Ulrich bundles. \square

Remark 6.7. Of course, the negative results obtained in this paper do not contradict the original conjecture. In [3] the authors show that most of the flag varieties in type A do not carry an irreducible Ulrich bundle. On the other hand, according to our experience, it is very limiting to restrict oneself to the class of irreducible bundles. There seem to be equivariant bundles that are not irreducible, but still enjoy incredibly nice properties (see, e.g., [6]). It is also interesting to see if our results can help construct Ulrich bundles on other varieties, e.g. those K uchle varieties that come from sections of Lagrangian Grassmannians of planes (see [8]).

REFERENCES

- [1] Arnaud Beauville. *Ulrich bundles on abelian surfaces*. Version 2. Dec. 7, 2015. arXiv:1512.00992v2 [math.AG].
- [2] Alexey Bondal and Mikhail Kapranov. “Homogeneous bundles”. In: *Helices and vector bundles*. Vol. 148. London Math. Soc. Lec. Note Series. Cambridge University Press, 1990, pp. 45–55.
- [3] Izzet Coskun et al. *Ulrich Schur bundles on flag varieties*. Version 1. Dec. 19, 2015. arXiv:1512.06193v1 [math.AG].
- [4] L. Costa and R. M. Mir -Roig. “GL(V)-invariant Ulrich bundles on Grassmannians”. In: *Mathematische Annalen* 361.1-2 (Feb. 2015), pp. 443–457. DOI: 10.1007/s00208-014-1076-9.
- [5] David Eisenbud, Frank-Olaf Schreyer, and Jerzy Weyman. “Resultants and Chow forms via exterior syzygies”. In: *J. Amer. Math. Soc.* 16.03 (July 2003), pp. 537–580. DOI: 10.1090/s0894-0347-03-00423-5.
- [6] A. V. Fonarev. “On the Kuznetsov-Polishchuk conjecture”. In: *Proceedings of the Steklov Institute of Mathematics* 290.1 (Aug. 2015), pp. 11–25. DOI: 10.1134/s0081543815060024.
- [7] J. Herzog, B. Ulrich, and J. Backelin. “Linear maximal Cohen-Macaulay modules over strict complete intersections”. In: *Journal of Pure and Applied Algebra* 71.2-3 (May 1991), pp. 187–202. DOI: 10.1016/0022-4049(91)90147-t.
- [8] A. G. Kuznetsov. “On K uchle varieties with Picard number greater than 1”. In: *Izvestiya: Mathematics* 79.4 (2015), pp. 698–709. DOI: 10.1070/IM2015v079n04ABEH002758.
- [9] Bernd Ulrich. “Gorenstein rings and modules with high numbers of generators”. In: *Mathematische Zeitschrift* 188.1 (Mar. 1984), pp. 23–32. DOI: 10.1007/bf01163869.
- [10] Jerzy Weyman. *Cohomology of vector bundles and syzygies*. Vol. 149. Cambridge University Press, 2003.

ALGEBRAIC GEOMETRY SECTION, STEKLOV MATHEMATICAL INSTITUTE, 8 GUBKIN STR., MOSCOW 119991
RUSSIA

E-mail address: `avfonarev@mi.ras.ru`